Subspaces:
If $U \subseteq V$, $\dim U \leq \dim V$
If $U \subseteq V$ and $\dim U = \dim V$, then $U = V$

Shortcuts:
If $\dim = n$, then L.I. list of length n is a basis, and spanning list of length n is a basis

Invertibility and Isomorphisms: 3C
T is invertible iff $\exists S$ s.t. $ST = I$, $TS = I$
A lin. map is invertible iff bijective
Isomorphism: An isomorphism is an invertible linear map.
Two vector spaces are isomorphic iff $\exists$ isomorphic map btw them
Two fin-dim vector spaces are isomorphic iff they have the same dimension

Dimension:
Any two bases of a fin-dim v.s. have the same length
$\dim V$ = length of any basis in V
$\dim \mathbb{F}^n = n$, $\dim P_m(\mathbb{F}) = m+1$ 2B

Quiz 2

Direct Sum:
$U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_k \in U_k\}$
Sum is smallest space containing each subspace
Direct: sum is direct if every elt in sum can be written in only one way
For two spaces $U + W$ is direct, denoted by $U \oplus W$ iff $U \cap W = \{0\}$, $\Rightarrow \dim(U \cap W) = 0$

Null Spaces and Ranges 3B
i) $\text{null } T = \{v \in V \mid Tv = 0\}$, $\text{null } T \subseteq \text{domain}$
T is injective $\iff \text{null } T = \{0\}$
ii) $\text{range } T = \{Tv \mid v \in V\}$, $\text{range } T \subseteq \text{codomain}$
T is surjective $\iff \text{range } T = W$
iii) FTLM: $\dim(\text{domain}) = \dim \text{null } T + \dim \text{range } T$
iv) $\dim V > \dim W$, then $\mathcal{L}(V,W)$ cannot be injective
$\dim V < \dim W$, then $\mathcal{L}(V,W)$ cannot be surjective

Linear Maps:
$T: V \to W$ satisfies: i) $T(u+v) = Tu + Tv$
ii) $T(\lambda v) = \lambda Tv$
$\mathcal{L}(V,W)$ is the set of all maps from $V \to W$
$\mathcal{L}(V,W)$ is a v.s. with addition, scalar mult
Zero map: Maps everything to 0
Identity map: Maps everything to itself
Thm: Can defn a map by specifying where basis vectors go
$\exists$ unique map s.t. $Tv_k = w_k$
Composition: $ST = S(T(v))$, typically $ST \neq TS$ 3A

3C: Matrices, Rank, Properties
Matrix of $T \in \mathcal{L}(V,W)$ with $B_V = \{v_1, \dots, v_n\}$ and $B_W = \{w_1, \dots, w_m\}$ is $A_T = A$
Set of $m \times n$ matrices $\mathbb{F}^{m,n}$ forms a v.s.
$T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$, then $M(ST) = M(S)M(T)$
Rank and Transpose:
- Column Rank: $\dim \text{span cols } A$
- Row Rank: $\dim \text{span rows } A$
- $A^T := (A^T)_{j,i} = A_{i,j}$

Ex: $A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 3 \end{pmatrix}$
In $\mathbb{F}^{2,1}$: $c_1 = \binom{1}{1}$, $c_2 = \binom{0}{1}$, $c_3 = \binom{-2}{3}$
Col Rank = 2 (vectors span $\mathbb{R}^2$, not $\mathbb{R}^3$)
In $\mathbb{F}^{1,3}$: $r_1 = [1,0,-2]$, $r_2 = [1,1,3]$
Row Rank = 2

Invertible: $A_{(n \times n)}$ is invertible if $\exists B$ s.t. $AB = I$, $A^{-1} = B$
Property: $(AC)^{-1} = C^{-1}A^{-1}$
Product Space: $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$
Summation Formula: $(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{r,k}$

3D: Isomorphisms, Coordinate Vectors and Rank, Change of Basis
lin map is invertible iff a bijection, ie an isomorphism
V, W are isomorphic iff $\dim V = \dim W$
$M: \mathcal{L}(V,W) \to \mathbb{F}^{m,n}$ is an isomorphism
$\dim \mathcal{L}(V,W) = \dim V \dim W$
Given basis $\{v_1, \dots, v_n\}$, any $v \in V$ written as $v = b_1 v_1 + \dots + b_n v_n$.
Also, coordinate matrix $M(v) = [b_1, \dots, b_n]^T$
Find coords of output: $M(Tv) = M(T)M(v)$
Properties: $\dim \text{null } T = \dim \text{null } M(T)$
$\dim \text{range } T = \text{col rank of } M(T)$
Change of Basis: I creates transition matrix $C = M(I, B_u, B_v)$
If A is A_T wrt B_u, and B is A_T wrt B_v, then $A = C^{-1}BC$

Ex: $v = \binom{2}{3}$ in $E = \{\binom{1}{0}, \binom{0}{1}\}$
In $B = \{\binom{1}{1}, \binom{-1}{1}\}$, find c_1, c_2 s.t. $c_1\binom{1}{1} + c_2\binom{-1}{1} = \binom{2}{3}$
Thus, $[v]_B = \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix}$

Quiz 3

3E: Quotient Spaces
Translates: For subspace $U \subseteq V$ and $v \in V$, $v + U = \{v + u \mid u \in U\}$
Equality: $v + U = w + U$ iff $v - w \in U$
Quotient Space: V/U is set of all translates $\{v + U \mid v \in V\}$ with:
- $(v + U) + (w + U) = (v + w) + U$
- $\lambda(v + U) = (\lambda v) + U$

Quotient Map: $\pi: V \to V/U$ defined by $\pi(v) = v + U$ is linear and surj.
Also, $\dim(V/U) = \dim V - \dim U$

Quiz Hints:
Q1: Build $M(T)$. Take each $e_i \in B$ and pass through T
Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$, it doubles x and flips y
$T(e_1) = 2e_1 + 0e_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
$T(e_2) = 0e_1 + -1e_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
$\Rightarrow M(T) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Q2: Check if a map is invertible (an isomorphism), you can check that $ST = I_V$, $TS = I_W$
Ex: $T = (x+y, x-y)$, find S that undoes it
Set $u = x+y$, $v = x-y$, then $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$
Define: $S(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$
Check: $S(T(x,y)) = S(x+y, x-y) = \left(\frac{(x+y)+(x-y)}{2}, \frac{(x+y)-(x-y)}{2}\right) = (x,y)$
$T(S(u,v)) = T\left(\frac{u+v}{2}, \frac{u-v}{2}\right) = (u,v)$
$\therefore ST = I$ and $TS = I$, T is an isomorphism.

Q3: Find basis of quotient space V/U
Steps: Start with B_W, extend for V, $B_{V/U} :=$ set of cosets $\{v_1 + U, \dots, v_n + U\}$
Ex: let $V = \mathbb{R}^3$, $U = z\text{-axis}$
$B_W = \{(0,0,1)\}$. $B_V = \{(0,0,1), (1,0,0), (0,1,0)\}$
$B_{V/U} = \{(1,0,0) + U, (0,1,0) + U\}$, hence why $\dim V/U = \dim V - \dim U = 3 - 1 = 2$

Def: *A linear functional is a linear map $V \to \mathbb{F}$, i.e., an element of $\mathcal{L}(V, \mathbb{F})$.*
The dual space of V is denoted by $V' = \mathcal{L}(V, \mathbb{F}) = \{\text{all linear functionals on } V\}$

Ex: *Any homogeneous degree -1 expression in n variables defines a linear functional on $\mathbb{F}^n$.*
$\varphi: \mathbb{R}^3 \to \mathbb{R}$, $\varphi(x,y,z) = 2x - y + 5z$

Ex: *Evaluation of polynomials and their derivatives:* $\varphi: \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, $\varphi(p(x)) = p(0) + p''(7)$

Prop: *If V finite dim, then $\dim V' = \dim V$.*
Pf: $\dim V' = \dim \mathcal{L}(V, \mathbb{F}) = \dim V \dim \mathbb{F} = \dim V$ ∎

Def: *Let $v_1, \dots, v_n$ be basis for V. The dual basis for V' is $\varphi_1, \dots, \varphi_n$ defined by:* $\varphi_i(v_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

Thm: *Let $v_1, \dots, v_n$ be basis for V. $\varphi_1, \dots, \varphi_n$ be dual basis for V'. Then:*
i) $\forall v \in V$, $v = \varphi_1(v)\cdot v_1 + \varphi_2(v)\cdot v_2 + \dots + \varphi_n(v) v_n$
ii) $\varphi_1, \dots, \varphi_n$ is a basis of V'

Pf: i) Write $v = c_1 v_1 + \dots + c_n v_n$. Then, $\varphi_i(v) = \varphi_i(c_1 v_1 + \dots + c_i v_i + \dots + c_n v_n)$
$= c_1 \varphi_i(v_1) + \dots + c_i \varphi_i(v_i) + \dots + c_n \varphi_i(v_n) = c_1 \varphi_i(v_1) = c_i$
"coordinate functions" on V wrt the basis
Thus $\forall c_i \in \mathbb{F}$, $c_i = \varphi_i(v)$. Considering $v = c_1 v_1 + \dots + c_n v_n$, then $\Leftrightarrow v = \varphi_1(v) v_1 + \dots + \varphi_n(v) v_n$, so (i) holds

ii) Claim: $\varphi_1, \dots, \varphi_n$ are l.i, then $c_1\varphi_1 + \dots + c_i\varphi_i + \dots + c_n\varphi_n = 0$
Plug in v_i: $(c_1\varphi_1 + \dots + c_i\varphi_i + \dots + c_n\varphi_n)(v_i) = 0(v_i)$ (zero functional)
$c_1\varphi_i(v_1) + \dots + c_i\varphi_i(v_i) + \dots + c_n\varphi_n(v_i) = 0$
$c_i \varphi_i(v_i) = 0$
$c_i = 0 \Rightarrow c_1 = \dots = c_n = 0$, thus l.c. was trivial
Thus, $\varphi_1, \dots, \varphi_n$ are independent, and with $n = \dim V = \dim V'$ length, $\therefore$ a basis ∎

Def: *Let $T \in \mathcal{L}(V, W)$. The dual map of T, $T' \in \mathcal{L}(W', V')$ is defined by:*
$(T'(\varphi))(v) = \varphi(Tv)$. Moreover, $T'\varphi = \varphi \circ T$
($\varphi \in W'$, $v \in V$; $\varphi \in W$)

Ex: *let* $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(a,b) = (2a+b, a-3b)$
$\varphi_1, \varphi_2 : (\mathbb{R}^2)'$; $\varphi_1, \varphi_2, \varphi_3 : (\mathbb{R}^3)'$
$T'\varphi_1(a,b) = \varphi_1(T(a,b)) = \varphi_1(2a+b,\ b,\ a-3b) = 2a+b = 2\varphi_1(a,b) + \varphi_2(a,b)$
$T'\varphi_1 = 2\varphi_1 + \varphi_2$

Prop: *Let $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$, then:*
i) $T' \in \mathcal{L}(W', V')$
ii) $(S+T)' = S' + T'$
iii) $(\lambda T)' = \lambda T'$
iv) $(ST)' = T'S'$ where $T \in \mathcal{L}(U, V)$, $S \in \mathcal{L}(V, W)$

Pf: $\forall \varphi$, $(ST)'(\varphi) = \varphi \circ ST = \varphi \circ S \circ T = T'(\varphi \circ S) = T'(S'(\varphi))$
$\therefore (ST)' = T'S'$ ∎

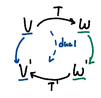
Def: *Let $U \subset V$ subspace, then the annihilator U^0 of U is:*
$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \ \forall u \in U\}$

Ex: $V = \mathbb{R}^2$. $U = \mathcal{L}(\{1,0\})$
$V' = \text{span}(\varphi_1, \varphi_2)$
If $\varphi = a\varphi_1 + b\varphi_2 \in U^0$, then $(a\varphi_1 + b\varphi_2)(1,0) = 0$
$a\varphi_1(1,0) + b\varphi_2(1,0) = a \cdot 1 + b \cdot 0 = 0$ ⇒ $a = 0$, thus $\varphi = b\varphi_2$
$\therefore U^0 = \text{span}(\varphi_2)$
$\mathbb{R}^2$: φ_1, φ_2; $\mathbb{R}^1$: φ_2

Prop:
i) U^0 is a subspace of V'
ii) $\dim U^0 = \dim V - \dim U$
iii) $U^0 = \{0\} \Leftrightarrow U = V$
iv) $U^0 = \{V'\} \Leftrightarrow U = \{0\}$

Prop: *Let V, W be fin dim, let $T \in \mathcal{L}(V, W)$. Then:*
i) $\begin{cases} \text{null } T' = (\text{range } T)^0 \\ \dim \text{null } T' = \dim \text{null } T + \dim W - \dim V \end{cases}$
ii) $\begin{cases} \dim \text{range } T' = \dim \text{range } T \\ \text{range } T' = (\text{null } T)^0 \end{cases}$

Corollary: T' is injective $\Leftrightarrow$ T is surjective
T' is surjective $\Leftrightarrow$ T is injective

Diagram:


Thm: *Let V, W fin dim, $T \in \mathcal{L}(V, W)$. $v_1, \dots, v_n : V$, $w_1, \dots, w_m : W$, $\varphi_1, \dots, \varphi_n : V'$, $\psi_1, \dots, \psi_m : W'$*
The matrix $M(T') = M(T)^T$
Pf: *Let $A = M(T)$, $B = M(T')$. WTS that $A_{i,j} = B_{j,i}$. Take* $Tv_j = A_{1,j}w_1 + \dots + A_{i,j}w_i + \dots + A_{m,j}w_m$
$T'\psi_i = B_{1,i}\varphi_1 + \dots + B_{j,i}\varphi_j + \dots + B_{n,i}\varphi_n$. Compute $(T'\psi_i)v_j$ as $(B_{1,i}\varphi_1 + \dots + B_{j,i}\varphi_j + \dots + B_{n,i}\varphi_n)v_j = B_{j,i} \cdot \psi_i(v_j) = B_{j,i}$
On the other hand $(T'\psi_i)v_j = \psi_i(Tv_j) = \psi_i(A_{1,j}w_1 + \dots + A_{i,j}w_i + \dots + A_{m,j}w_m) = A_{i,j}$, thus $A_{i,j} = B_{j,i}$

3 Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Since $v \neq 0$, $\{v\}$ is l.i. Extending to a basis in V: $\{v, v_2, \dots, v_n\}$.
Define $\varphi: V \to \mathbb{F}$ s.t. $\varphi(v) = 1$ and $\varphi(v_i) = 0$ for $i \in \{2, \dots, n\}$
$\forall u \in V$, $u = c_1 v + c_2 v_2 + \dots + c_n v_n$
$\varphi(u) = c_1\varphi(v) + c_2\varphi(v_2) + \dots + c_n\varphi(v_n)$
$= c_1 \cdot 1 + c_2 \cdot 0 + \dots + c_n \cdot 0 = c_1$
By construction, $\exists \varphi \in V'$ s.t. $\varphi(v) = 1$

4 Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

A basis for U is $\{u_1, \dots, u_n\}$ with $\dim U = n$
We can construct a basis for V by extension: $\{u_1, \dots, u_n, v_1, \dots, v_m\}$ for $\dim V = n+m$
Define $\varphi: V \to \mathbb{F}$ where $\varphi \in V'$, s.t. $\varphi(u_i) = 1$, $\varphi(v_i) = 0$ and $\varphi(\omega) = 0$ for $i \in \{2, \dots, n\}$ and thus
then for $u \in U$, $u = c_1 u_1 + \dots + c_n u_n$
$\varphi(u) = c_1\varphi(u_1) + \dots + c_n\varphi(u_n) = 0 + \dots + 0 = 0$
but for $v \in V$, $v = c_1 u_1 + \dots + c_n u_n + c_{n+1} v_1 + \dots + c_{n+m} v_m$
$\varphi(v) = c_1 \cdot 0 + \dots + c_n \cdot 0 + c_{n+1} \cdot 1 + \dots + c_{n+m} \cdot 0$
$= c_{n+1}$. Thus, $\varphi(v) = 0$ $\forall u \in U$ but $\varphi \neq 0$ by construction

6 Suppose $\varphi, \beta \in V'$. Prove that $\text{null } \varphi \subseteq \text{null } \beta$ if and only if there exists $c \in \mathbb{F}$ such that $\beta = c\varphi$.

$\Rightarrow$: Suppose $\text{null}\varphi \subseteq \text{null}\beta$
If $\varphi = 0$, then $\text{null}\,\varphi = V$, so $\text{null}\,\beta = V$ too.
This means $\beta = 0$ so $c = 0$
If $\varphi \neq 0$, then we know $\exists v \in V$ s.t. $\varphi(v) = 1$
For any $w \in V$, $w = w - \varphi(w)v + \varphi(w)v$
Let $u = w - \varphi(w)v$, then $\varphi(u) = \varphi(w) - \varphi(w)\varphi(v) = 0$, so $u \in \text{null}\,\varphi$
thus $u \in \text{null}\beta$ too, i.e., $\beta(u) = 0$
β to w: $\beta(w) = \beta(u) + \varphi(w)\beta(v)$
$= 0 + \varphi(w)\beta(v)$
Let $c = \beta(v)$, then $\beta(w) = c\varphi(w)$, i.e. $\beta = c\varphi$
$\Leftarrow$: $\beta = c\varphi$. Then let $v \in \text{null}\,\varphi$, $\varphi(v) = 0$
Then, $\beta(v) = (c\varphi)(v) = c\varphi(v) = c \cdot 0 = 0$, so $v \in \text{null}\beta$ too $\Rightarrow \text{null}\,\varphi \subseteq \text{null}\beta$.

9 Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \dots, x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}$$

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

WTS that $\varphi_k(x^j) = \begin{cases} 1 \text{ if } k = j \\ 0 \text{ if } k \neq j \end{cases}$
If $k > j$, $(x^j)^{(k)} = 0$
If $k < j$, $(x^j)^{(k)} = \frac{j!}{(j-k)!} x^{j-k}$. At $x = 0$, $\varphi_k(x^j) = 0$
If $k = j$, then $(x^k)^{(k)} = k!$. At $x = 0$, $k!/k! = 1$

13 Show that the dual map of the identity operator on V is the identity operator on V'.

16 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \iff T = 0.$$

28 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim((\text{null}\,\varphi_1) \cap \cdots \cap (\text{null}\,\varphi_m)) = (\dim V) - m.$$

Suppose $\dim V = m$. Suppose $c_1\varphi_1 + \dots + c_m\varphi_m = 0$ s.t. $c_1 = \dots = c_m = 0$
Define $\mathcal{T}\!\mathcal{V} = (\varphi_1, \dots, \varphi_m V)$, and $\mathcal{T}\!\mathcal{V} \in \mathcal{L}(V, \mathbb{F}^m)$
$\text{null}\,\mathcal{T}\!\mathcal{V} = \text{null}\,\varphi_1 \cap \dots \cap \text{null}\,\varphi_m$
$\text{range}\,\mathcal{T}\!\mathcal{V} = \mathbb{F}^m$
By RNT, $\dim(\text{null}\,\mathcal{T}\!\mathcal{V}) + \dim(\text{range}\,\mathcal{T}\!\mathcal{V}) = \dim V$
$\dim(\text{null}\,\varphi_1 \cap \dots \cap \text{null}\,\varphi_m) + \dim(\mathbb{F}^m) = \dim V$
$\dim(\text{null}\,\varphi_1 \cap \dots \cap \text{null}\,\varphi_m) = \dim V - m$

6 Suppose $\varphi, \beta \in V'$. Prove that $\text{null}\,\varphi \subseteq \text{null}\,\beta$ if and only if there exists $c \in \mathbb{F}$ such that $\beta = c\varphi$.

$\Rightarrow$: Suppose $\text{null}\,\varphi \subseteq \text{null}\,\beta$
Then if $\varphi = 0$, then $\text{null}\,\varphi \subseteq V$. So $\text{null}\beta = V$ too
so $\beta = 0$ and $c = 0$
if $\varphi \neq 0$, $\exists v \in V$ s.t. $\varphi(v) = 1$
Let $w \in V$, $w = w - \varphi(w)v + \varphi(w)v$
take $u = w - \varphi(w)v$ Then $\varphi(u) = \varphi(w) - \varphi(w)\varphi(v) = \varphi(w) - \varphi(w) = 0$
so, $u \in \text{null}\,\varphi$, then by $\text{null}\,\varphi \subseteq \text{null}\beta$, $u \in \text{null}\beta$, so $\beta(u) = 0$
back to w, $\beta(w) = \beta(u) + \varphi(w)\beta(v) = \varphi(w)\beta(v)$
Let $c = \beta(v)$, so we have $\beta = c\varphi$
$\Leftarrow$: Suppose $\exists c \in \mathbb{F}$ s.t. $\beta = c\varphi$
Then let $v \in \text{null}\,\varphi$, $\varphi(v) = 0$
Then, $\beta(v) = c\varphi(v) = c\varphi(v) = 0 \Rightarrow v \in \text{null}\beta$
thus, $\text{null}\,\varphi \subseteq \text{null}\beta$

Minimal Polynomial:

Definitions:

- Monic: monic poly. has highest-deg. coeff. = 1
  Ex: $2+9z^2+z^7$ is monic poly. of deg. 7
- $T\in\mathcal{L}(V)$, V fin-dim. Minimal polynomial of T is the unique monic poly. $p\in P(\mathbb{F})$ of smallest degree s.t. $p(T)=0$
  Ex: $p(z)=2-c$ is monic deg 1. If $p(T)=0$, then $T=cI$

Propositions/Theorems:

- Every operator on fin-dim complex v.s. has an eigenvalue
- $T\in\mathcal{L}(V)$, V fin-dim. Then $\exists$ unique monic poly. $p\in P(\mathbb{F})$ of smallest degree s.t. $p(T)=0$, and $\deg p\le \dim V$.
- Eigenvalues = zeros of min. poly.: Suppose $T\in\mathcal{L}(V)$, V fin-dim:
  i) zeros of the min. poly. of T are the eigenvalues of T
  ii) If V is complex v.s., min poly. of T: $(z-\lambda_1)\dots(z-\lambda_m)$ where $\lambda_1,\dots,\lambda_m$ are eigenvalues of T
- $q(T)=0 \iff q$ is a poly. multiple of the min. poly.
- If $U\subset V$ subspace invariant under T, min poly of T is a poly. multiple of the min. poly. of $T|_U$
- T not invertible $\iff$ constant term of min. poly. is 0
- if $b^2<4c$, $\dim\operatorname{null}(T^2+bT+cI)$ is an even number
- every operator on an odd-dim v.s. has an eigenvalue

4 Suppose $\mathbf{F}=\mathbf{C}$, $T\in\mathcal{L}(V)$, $p\in\mathcal{P}(\mathbf{C})$ is a nonconstant polynomial, and $a\in\mathbf{C}$. Prove that a is an eigenvalue of $p(T)$ if and only if $a=p(\lambda)$ for some eigenvalue λ of T.

$\Longleftarrow$: Suppose $a=p(\lambda)$ for some eigenvalue λ of T.
WTS that if $Tv=\lambda v$, then $p(T)v=p(\lambda)v=av$
$p(\lambda)=a_n\lambda^n+\dots+a_1\lambda+a_0$
$p(T)v=a_nT^nv+\dots+a_0Iv$
$=a_n\lambda^nv+\dots+a_0v$
$=v(a_n\lambda^n+\dots+a_1\lambda+a_0)$
$=p(\lambda)v=av$

$\Longrightarrow$: Suppose $p(T)v=av$, WTS $a=p(\lambda)v$ where $v\neq0$
$v(p(T)-aI)=0$. Define $f(z)=p(z)-a$, where $f(T)$ not inj. since maps $v\neq0$ to 0
by FTLA, $f(T)=c(z-\lambda_1)\dots(z-\lambda_n)$
then, $v(T-\lambda_1)\dots(T-\lambda_n)c=0$. Since $f(T)$ not inj., then not invertible, thus one of $T-\lambda_kI$ is not invertible
Thus, $(T-\lambda_kI)$ has λ_k eigenvalue of T, so λ_k root to $f(z)$
Then, $f(\lambda_k)=0$
$p(\lambda_k)-a=0$
$p(\lambda_k)=a$ for some λ_k ■

11 Suppose V is a two-dimensional vector space, $T\in\mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix}a&c\\b&d\end{pmatrix}$.

(a) Show that $T^2-(a+d)T+(ad-bc)I=0$.
(b) Show that the minimal polynomial of T equals

$$\begin{cases}z-a & \text{if } b=c=0 \text{ and } a=d,\\ z^2-(a+d)z+(ad-bc) & \text{otherwise.}\end{cases}$$

a) $\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}a&c\\b&d\end{pmatrix}=\begin{pmatrix}a^2+cb & ac+cd\\ ab+bd & bc+d^2\end{pmatrix}$ $\quad\begin{pmatrix}a^2+cb & ac+cd\\ ab+bd & bc+d^2\end{pmatrix}-\begin{pmatrix}a^2+cd & ac+cd\\ ab+bd & ad+d^2\end{pmatrix}+\begin{pmatrix}ad-bc & 0\\0 & ad-bc\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=0$
$(a+d)\begin{pmatrix}a&c\\b&d\end{pmatrix}=\begin{pmatrix}a^2+cd & ac+cd\\ ab+bd & ad+d^2\end{pmatrix}$
$(ad-bc)\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}ad-bc & 0\\0 & ad-bc\end{pmatrix}$

b) if $b=c=0$, $a=d$, then $T=\begin{pmatrix}a&c\\b&d\end{pmatrix}=\begin{pmatrix}a&0\\0&0\end{pmatrix}=a\begin{pmatrix}1&0\\0&1\end{pmatrix}$ where $q(z)=z-a$
other way, $T-\lambda I=0\Rightarrow T=\lambda I$, $\begin{pmatrix}a&c\\b&d\end{pmatrix}=\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}$, thus $b=c=0$, $a=d=\lambda$
otherwise, ------,

20 Suppose $T\in\mathcal{L}(\mathbf{F}^3)$ is such that the eigenvalues of T are $3,5,8$. Prove that $(T-3I)^2(T-5I)^2(T-8I)^2=0$.

Pf: zeros of min. poly. are eigenvalues of T
Since 3,5,8 are eigenvalues, write $p(z)=(z-3)^{m_1}(z-5)^{m_2}(z-8)^{m_3}$
$\deg p\le\dim V\Rightarrow m_1+m_2+m_3\le4$, so m_j is at most 2
then if we write $(T-3I)^2(T-5I)^2(T-8I)^2$, then this is a multiple of min. poly. $p(z)$, thus $=0$
$\therefore(T-3I)^2(T-5I)^2(T-8I)^2=0$ ■

21 Suppose V is finite-dimensional and $T\in\mathcal{L}(V)$. Prove that the minimal polynomial of T has degree at most $1+\dim\operatorname{range}T$.

If $\dim\operatorname{range}T<\dim V-1$, then this exercise gives a better upper bound than 5.22 for the degree of the minimal polynomial of T.

We know $\deg p\le\dim V$.
If we def. $T|_{\operatorname{range}T}$ with $q(T)|_{\operatorname{range}T}=0$, then $qT\cdot Tv=0$ and $\deg q\le\dim\operatorname{range}T$

$q(T)(Tv)=T(q(T))v$ with poly. $f(z)=zq(z)$ of degree $1+\deg q$
since $\deg q\le\dim\operatorname{range}T$, then $\deg f$ is at most $1+\dim\operatorname{range}T$
Thus, $\deg p\le\deg f\le1+\dim\operatorname{range}T$

7 (a) Give an example of $S,T\in\mathcal{L}(\mathbf{F}^2)$ such that the minimal polynomial of ST does not equal the minimal polynomial of TS.
(b) Suppose V is finite-dimensional and $S,T\in\mathcal{L}(V)$. Prove that if at least one of S,T is invertible, then the minimal polynomial of ST equals the minimal polynomial of TS.
Hint: Show that if S is invertible and $p\in\mathcal{P}(\mathbf{F})$, then $p(TS)=S^{-1}p(ST)S$.

a) $S(e_1)=0$, $S(e_2)=e_1=(1,0)$: $S=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $ST(e_1)=S(e_1)=0$, $ST(e_2)=S(0)=0$, $q(z)=z$
$T(e_1)=e_1$, $T(e_2)=(0,0)$: $T=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $TS(e_1)=T(0)=0$, $TS(e_2)=T(e_1)=e_1$, $(TS)^2e_1=TS(e_1)=0$, $q(z)=z^2$, so $z\neq z^2$
b) Hint: $(TS)^k=TS\cdot\underline{A}\cdot TS=T(ST)\cdots(ST)S=T(ST)^{k-1}S=S^{-1}(ST)ST^{k-1}S=S^{-1}ST^kS$
$p(TS)=a_n(TS)^n+\dots+a_0I$
$=a_n(S^{-1}(ST)^nS)+\dots+a_0I$
$=S^{-1}(a_n(ST)^n+\dots+a_0I)S$
$=S^{-1}p(ST)S$
Pf: $q_{ST}(ST)=0$, $q(TS)=S^{-1}q(ST)S=0\Rightarrow\deg q_{TS}\le\deg q_{ST}$
$q_{TS}(TS)=0$, $q_{TS}(ST)=Sq(TS)S^{-1}=0\Rightarrow\deg q_{ST}\le\deg q_{TS}\Rightarrow q_{ST}=q_{TS}$

17 Suppose V is finite-dimensional, $T\in\mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda\in\mathbf{F}$. Show that the minimal polynomial of $T-\lambda I$ is the polynomial q defined by $q(z)=p(z+\lambda)$.

$p(T)=0$. $q(z)=p(z+\lambda)$
$q(T-\lambda I)=p((T-\lambda I)+\lambda I)$
$=p(T)$, so $q(T-\lambda I)=0$
Suppose s has smaller deg s.t. $s(T-\lambda I)=0$.
Then def. $r(z)=s(z-\lambda)$. With T, $r(T)=s(T-\lambda I)=0$
but $p(T)=0$ with p smallest deg, so r D.N.E $\Rightarrow$ s D.N.E, so q is min. poly. ■

19 Suppose V is finite-dimensional and $T\in\mathcal{L}(V)$. Let $\mathcal{E}$ be the subspace of $\mathcal{L}(V)$ defined by

$$\mathcal{E}=\{q(T):q\in\mathcal{P}(\mathbf{F})\}.$$

Prove that $\dim\mathcal{E}$ equals the degree of the minimal polynomial of T.

WTS $\{I,T,\dots,T^{n-1}\}$ basis of $\mathcal{E}$, where $\deg p=n$ s.t. $p(T)=0$
Suppose $a_{n-1}T^{n-1}+\dots+a_0I=0$, then $f(z)=a_{n-1}z^{n-1}+\dots+a_0$ s.t. $f(T)=0$
We know $\exists p$ s.t. $p(T)=0$ with deg n. so $f(T)=0$ only if $a_0=\dots=a_{n-1}=0$, thus set is l.i.
Next, $\forall q(T)\in\mathcal{E}$, written as lc of $\{I,T,\dots,T^{n-1}\}$
$\forall q(z)$, write $q(z)=s(z)p(z)+r(z)$ by division algo. and where $\deg r<\deg p=n$
$q(T)=s(T)p(T)+r(T)=r(T)$ since $p(T)=0$
because $\deg r\le n-1$, its lc of $\{I,T,\dots,T^{n-1}\}$, thus spans $\mathcal{E}$
Thus $\{I,T,\dots,T^{n-1}\}$ basis for $\mathcal{E}$ with $\dim\mathcal{E}=n=\deg p$ ■

Invariant Subspaces And Upper-Triangular Matrices:

Definitions:

- An operator on v.s. V is a map $T \in \mathcal{L}(V)$
- $T \in \mathcal{L}(V)$. $U \subseteq V$ subspace is T-invariant if $Tu \in U$ $\forall u \in U$ $\left(T|_U : U \to U\right)$
- $\lambda \in \mathbb{F}$ is an eigenvalue of T if $\exists v \neq 0 \in V$ s.t. $Tv = \lambda v$. v is the corresponding eigenvector
- $p(z) = a_n z^n + \dots + a_1 z + a_0$, then $p(T) = a_n T^n + \dots + a_1 T + a_0 I$. Also, $p(T)v \in V$
- $\forall p$, $p(T)v = pv$. Ex: $(T-2I)v = (T-2)v$
- **Matrix of Operator:** $T \in \mathcal{L}(V)$, $B_V = \{v_1, \dots, v_n\}$, $M(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$ and entries defined by $Tv_k = A_{1,k}v_1 + \dots + A_{n,k}v_n$
- **Ex:** $T \in \mathcal{L}(\mathbb{F}^3)$, $T(x,y,z) = (2x+y, 5y+3z, 8z)$
  $M(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$, then eigenvalues are 2, 5, 8 by theorem
- **Upper-Diagonal Matrix:** A square matrix is upper-triangular if all entries below diagonal = 0
- Visual: $\begin{pmatrix} \lambda_1 & * \\ & \ddots \\ 0 & \lambda_n \end{pmatrix}$, where * reps. irrelevant/unknown entries

Propositions:

- Let V fin-dim. Let $T \in \mathcal{L}(V)$. Let $\lambda \in \mathbb{F}$. TFAE:
  a) λ is an eigenvalue of T
  b) $T - \lambda I$ is not injective
  c) $T - \lambda I$ is not surjective
  d) $T - \lambda I$ is not invertible
  (b), (c), (d) automatically equivalent
- $\text{null}(T - \lambda I) \setminus \{0\}$ is the set of all eigenvectors with eigenvalue λ
- V be fin-dim $= n$. $T \in \mathcal{L}(V)$. Then T has at most n different eigenvectors.
- If $T \in \mathcal{L}(V)$, then $p(T) \in \mathcal{L}(V)$, ie linear
- Let $T \in \mathcal{L}(V)$, $p(z) \in \mathbb{P}(\mathbb{F})$, then $\text{null}(p(T))$, $\text{range}(p(T))$ are T-invariant subspaces
  Pf Sketch: Let $p(T)v = 0$. $p(T)(Tv) = T(p(T)v)$, thus $T(0) = 0$ and $Tv \in \text{null}(p(T))$
  Let $p(T)u = w$, then $Tw = T(p(T)u) = p(T)Tu \in \text{range}(p(T))$
- T has up.-tri. matrix wrt some B_V iff min. poly. $= (z-\lambda_1)\cdots(z-\lambda_m)$ with λ_i's $\in \mathbb{F}$
- If $\mathbb{F} = \mathbb{C}$, then every operator has up.-tri. matrix

Theorems:

- any set of eigenvectors of T with distinct eigenvalues is lin indep.
- Let V be fin-dim over $\mathbb{C}$, $T \in \mathcal{L}(V)$, then T has an eigenvalue
- **Minimal Poly.:** Let V v.s. fin dim over $\mathbb{F}$, Let $T \in \mathcal{L}(V)$. Then $\exists$ unique monic poly. of smallest degree s.t. $p(T) = 0$
  then p is the minimal poly. of T and $\deg p \leq \dim V$
- **Upper-Triangular Characterization:** $T \in \mathcal{L}(V)$, $B_V = \{v_1, \dots, v_n\}$, TFAE:
  i) $M(T, B_V)$ is upper-triangular
  ii) $\text{span}(v_1, \dots, v_k)$ is invariant under T $\forall k \in [1, n]$
  iii) $Tv_k \in \text{span}(v_1, \dots, v_k)$ $\forall k \in [1, n]$
- $T \in \mathcal{L}(V)$, B_V exists s.t. T has up-tri. matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then $(T-\lambda_1 I)\cdots(T-\lambda_n I) = 0$
- If T has up.-tri. matrix wrt some B_V, then eigenvalues of T = diagonal entries

1, 3, 7, 8, 9, 12a, 13

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2 Suppose that $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

Pf: $\forall i$, $\forall v \in V_i$, $Tv_j \in V_i$ $\forall j$
Let $v \in V_1 + \dots + V_m$, then $v = v_1 + \dots + v_m$
$Tv = T(v_1 + \dots + v_m) = Tv_1 + \dots + Tv_m$
Since V_i is T-invariant, $Tv_i \in V_i$, thus $Tv_1 + \dots + Tv_m \in V_1 + \dots + V_m$
$\Rightarrow Tv \in V_1 + \dots + V_m$ ∎

3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

know that $\forall i$, $Tv_i \in V$; $\forall v_i \in V$
WTS that $V_1 \cap \dots \cap V_m$ is T-inv., ie $Tv \in V_1 \cap \dots \cap V_m$ $\forall v \in V_1 \cap \dots \cap V_m$, $v = (v_1, \dots, v_m)$
Pf:
Let $v \in V_1 \cap \dots \cap V_m$, then $v \in V_j$ $\forall j \in [1, m]$
Thus, $Tv \in V_j$ since V_j is T-inv.
Since Tv in all V_j, Tv in their intersection,
ie $Tv \in V_1 \cap \dots \cap V_m$ ∎

1 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and T^2 has an upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some basis of V.

Let $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $T^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is upper-triangular
$T - \lambda I = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$. $\det(T - \lambda I) = \lambda^2 + 1$, which has complex λ
$\therefore$ no basis in $\mathbb{R}^2$ that makes T up.-tr., thus proven false.

3 Suppose $T \in \mathcal{L}(V)$ is invertible and $v_1, \dots, v_n$ is a basis of V with respect to which the matrix of T is upper triangular, with $\lambda_1, \dots, \lambda_n$ on the diagonal. Show that the matrix of T^{-1} is also upper triangular with respect to the basis $v_1, \dots, v_n$, with

$$\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$$

on the diagonal.

$T = \begin{pmatrix} \lambda_1 & * \\ & \ddots \\ 0 & \lambda_n \end{pmatrix}$, and $Tv_k \in \text{span}(v_1, \dots, v_k) = U_k \Rightarrow T(U_k) \subseteq U_k$
Because T is invertible and maps $T(U_k)$ to U_k,
then $\Rightarrow T(U_k) = U_k$.
Then, by invertibility, $U_k = T^{-1}(U_k)$, thus T^{-1} wrt $v_1, \dots, v_n$ is up.-tri.
Then, $M(T^{-1})$ and $M(T)$ are up.-tri, and $M(T^{-1})_{k,j} M(T)_{j,k} = I$ is up.-tri.
$M(T^{-1})M(T) = M(T^{-1})\lambda_k = 1 \Rightarrow M(T^{-1}) = 1/\lambda_k$.

7 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$.
(a) Prove that there exists a unique monic polynomial p_v of smallest degree such that $p_v(T)v = 0$.
(b) Prove that the minimal polynomial of T is a polynomial multiple of p_v.

a) $\forall v \in V$, the list $v, Tv, T^2v, \dots, T^n v$ is linearly dependent
because V is fin-dim with $\dim V = n$. Thus, nonzero c_i's exist
s.t. $c_0 v + c_1 Tv + \dots + c_n T^n v = 0$. Equivalently, there exists
some nonzero polynomial $q(z) = c_0 + c_1 z + \dots + c_n z^n$ s.t. $q(T) = 0$
Then, since at least one polynomial exists, there exists one of smallest degree.
Let this be $p_v(z)$ with leading coeff. a, then we define $p_v(z) = \frac{1}{a} q(z)$, where
p_v is monic of smallest degree s.t. $p_v(T)v = 0$
If we have two min. poly p_v and g of deg. d, then $p_v Tv = 0 = gTv$
Define $h = p_v - g$, where the z^d term cancels on RHS, resulting in $\deg(h) < d$
But, d is smallest degree, thus $\deg(h)$ must be 0 and h is not nonzero $\Rightarrow 0 = p_v - g \Rightarrow p_v = g$ ∎
b) Let p be min poly, so $p(T) = 0$. $p(T)u = 0$ $\forall u \in V$, and for v above, $pTv = 0$
Then, $p(z) = p_v(z)q(z) + r(z)$ with $r(z) = 0$ or $\deg r < \deg p_v$
$pTv = (p_v T \cdot qT)v + rTv$
$0 = qT \cdot 0 + rTv$, so $r(z) = 0$, thus $p(z) = p_v(z)q(z)$, ie a poly. multiple ∎

8 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and there exists a nonzero vector $v \in V$ such that $T^2 v + 2Tv = -2v$.
(a) Prove that if $\mathbb{F} = \mathbb{R}$, then there does not exist a basis of V with respect to which T has an upper-triangular matrix.
(b) Prove that if $\mathbb{F} = \mathbb{C}$ and A is an upper-triangular matrix that equals the matrix of T with respect to some basis of V, then $-1 + i$ or $-1 - i$ appears on the diagonal of A.

a) if $\mathbb{F} = \mathbb{R}$, eigenvalues λ of $T \in \mathbb{R}$.
$T^2 v + 2Tv + 2v = 0$
$v(T^2 + 2T + 2) = 0$, which has complex roots, so eigenvalues of p_v s.t. $p(T)v = 0$ are not in the field $\mathbb{R}$
$\therefore$ d.n.e. basis to which T is upper triangular
b) take $(T^2 + 2T + 2I)v = 0$ again, and eigenvalues $\lambda \in \mathbb{C}$
then, $\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i \Rightarrow \lambda_1 = i - 1$, $\lambda_2 = -1 - i$

We know that the diagonal entries of an upper-triangular matrix are the eigenvalues.
Since v is nonzero in the kernel of $T^2 + 2T + 2I$, at least one root of $P(x)$ is an eigenvalue of T,
i.e., a diagonal entry of the upper triangular matrix of T

9 Suppose B is a square matrix with complex entries. Prove that there exists an invertible square matrix A with complex entries such that $A^{-1}BA$ is an upper-triangular matrix.

12 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V, and U is a subspace of V that is invariant under T.
(a) Prove that $T|_U$ has an upper-triangular matrix with respect to some basis of U.

13 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose there exists a subspace U of V that is invariant under T such that $T|_U$ has an upper-triangular matrix with respect to some basis of U and also T/U has an upper-triangular matrix with respect to some basis of V/U. Prove that T has an upper-triangular matrix with respect to some basis of V.

8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

$Pv = \lambda v$
$P(Pv) = P(\lambda v)$
$P^2 v = \lambda P(v)$
$Pv = \lambda \lambda v$
$\lambda v = \lambda^2 v$ holds only if $\lambda = \{0, 1\}$

10 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigenvalues and eigenvectors of T.

$(Tp)(x) = xp'(x)$, find all λ s.t.
$(Tp)(x) = xp'(x) = \lambda p(x)$
test with basis of $\mathbb{P}_4(x)$: $\{1, x, x^2, x^3, x^4\}$
$x \cdot 0 = 0 \cdot x$, $\lambda = 0$
$x \cdot 1 = 1 \cdot x$, $\lambda = 1$
$x \cdot 2x = 2x^2 = \lambda p(x)$, $\lambda = 2$
$x \cdot 3x^2 = 3x^3$, $\lambda = 3$
$x \cdot 4x^3 = 4x^4$, $\lambda = 4$
$\lambda = \{0, 1, 2, 3, 4\}$

13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.
(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.
(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Suppose $Tv = \lambda v$, WTS that $S^{-1}TSv = \lambda v$
$S^{-1}TSS^{-1}v = S^{-1}Tv = S^{-1}\lambda v = \lambda S^{-1}v$, so λ eigenvec to $S^{-1}TS$
b) $S^{-1}TS$ is like a change of basis, but λ stays the same

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

$Tu = \lambda_1 u$, $Tw = \lambda_2 w$, $T(u+w) = \lambda_3(u+w)$
$T(u+w) = T(u) + T(w) = \lambda_1 u + \lambda_2 w = \lambda_3(u+w) = \lambda_3 u + \lambda_3 w$
then, $u(\lambda_1 - \lambda_3) + w(\lambda_2 - \lambda_3) = 0$, so $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_3$

30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Let $v \in V$, then $(T-2I)(T-3I)(T-4I)v = 0v = 0$
by $p(T)v = \lambda v$, we have $(\lambda-2)(\lambda-3)(\lambda-4) = 0 \Rightarrow \lambda = \{2, 3, 4\}$

Diagonalizable Operators And Commuting Operators:

E 2.3, 5, 6, 9a, 10

Definitions:

- **Diagonalizable:** operator V is diagonalizable if V has diagonal matrix wrt some B_V
- **Eigenspace:** Eigenspace of $T \in \mathcal{L}(V)$ corresponding to λ is the subspace $E(\lambda,T)$ of V defined by $E(\lambda,T) = \text{null}(T-\lambda I) = \{v \in V : Tv = \lambda v\}$. Hence $E(\lambda,T)$ is set of all eigenvectors and $\vec{0}$
- **Ex:** $M(T) = \begin{pmatrix} 8 & * \\ 0 & 5 \end{pmatrix}$, $T \in \mathcal{L}(V)$ wrt basis v_1, v_2, v_3, then $E(8,T) = \text{span}(v_1)$, $E(5,T) = \text{span}(v_2, v_3)$
- **Commute:** S, T on same vector space commute if $ST = TS$
  A, B square matrices of same size commute if $AB = BA$
  Also, S, T commute iff $M(S, B_V)$ and $M(T, B_V)$ commute

Propositions/Theorems:

- If λ eigenvalue of $T \in \mathcal{L}(V)$, then $T|_{E(\lambda,T)} = \lambda v \ \forall v \in V$
- Sum of eigenspaces is direct, and dim of each eigenspace $\leq \dim V$
- **Diagonalizability Characterization:** $T \in \mathcal{L}(V)$. V finitedim, $\lambda_1, \dots, \lambda_m$ eigenvalues of T, TFAE:
  i) T is diagonalizable
  ii) V has a basis consisting of eigenvectors of T
  iii) $V = E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T)$
  iv) $\dim V = \dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T)$
- **Enough Eigenvalues Implies Diagonalizability:**
  If V finitedim and $T \in \mathcal{L}(V)$ has $\dim V$ eigenvalues, then T is diagonalizable
- **Condition for Diagonalizability:** If V finitedim and $T \in \mathcal{L}(V)$,
  then T is diagonalizable iff min poly of $T = (z-\lambda_1)\cdots(z-\lambda_m)$ for some distinct $\lambda_1, \dots, \lambda_m \in \mathbb{F}$
- Suppose $T \in \mathcal{L}(V)$ diagonalizable and $U \subseteq V$ T-inv., then $T|_U$ diagonalizable

5.75: $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbb{F}$, then $E(\lambda,S)$ is invariant under T

5.76: Two diagonalizable operators on the same v.s. have diag. matrices wrt same basis iff the two operators commute

5.78: Every pair of commuting operators on finitedim nonzero complex v.s. has a common eigenspace

5.80: V finitedim complex v.s., and S, T commute on V, then $\exists B_V$ wrt which both S, T have upper-tri. matrices

5.81: Suppose V finitedim complex v.s. and S, T commute, then:
- every e-val of $S+T$ is an e-val of S plus an e-val of T
- every e-val of ST is an e-val of S times an e-val of T

---

2 Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V. Prove that if $\lambda \in \mathbf{F}$, then λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

$E(\lambda,T) = \text{null}(T-\lambda I)$

# of occurrences of λ = # of zeroes in diagonal of $M(T)-\lambda I$
$= \dim \text{null}(T-\lambda I)$
$= \dim E(\lambda,T)$

Ex: $M(T) = \begin{pmatrix} 3 & * \\ 0 & 2 \end{pmatrix}$
$T - 2I = \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}$
$0x_1 = 0$. x_1 free. $3x_2 = 0$, $x_2 = 0$. $0x_3 = 0$, x_3 free
$\text{null}(T-2I) = \{(x_1, 0, x_3) \mid x_1, x_3 \in \mathbb{F}\}$
$\dim \text{null}(T-2I) = 2$ as required

4 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.
(a) $V = \text{null } T \oplus \text{range } T$.
(b) $V = \text{null } T + \text{range } T$.
(c) $\text{null } T \cap \text{range } T = \{0\}$.

TFAE proof.
a) $\Rightarrow$ b) is clear
b) $\Rightarrow$ c) if $V = \text{null}T + \text{range}\,T$
$\dim V = \dim(\text{null}\,T + \text{range}\,T)$
$= \dim \text{null}\,T + \dim \text{range}\,T - \dim(\text{null}\,T \cap \text{range}\,T)$
Also, by RNT, $\dim V = \dim \text{null}\,T + \dim \text{range}\,T \Rightarrow \text{null}\,T \cap \text{range}\,T = \{0\}$
c) $\Rightarrow$ a) Since $\text{null}\,T \cap \text{range}\,T = \{0\}$, $\text{null}\,T + \text{range}\,T$ is direct
so $\dim(\text{null}\,T \oplus \text{range}\,T) = \dim \text{null}\,T + \dim \text{range}\,T = \dim V$

9 Suppose $R, T \in \mathcal{L}(\mathbf{F}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbf{F}^3)$ such that $R = S^{-1}TS$.

$Rv_1 = 2v_1$, $Rv_2 = 6v_2$, $Rv_3 = 7v_3$ since $\dim \mathbb{F}^3 = 3$, thus $\dim E(\lambda_i, T) = 1 \ \forall i \in \{1,2,3\}$
$Tw_1 = 2w_1$, $Tw_2 = 6w_2$, $Rw_3 = 7w_3$
Define S as $Sv_i = w_i$
So $S^{-1}TSv_1 = S^{-1}T(Sv_1) \cdots$ Thus $R = S^{-1}TS$
$= S^{-1}Tw_1$,
$= S^{-1}2w_1$,
$= 2S^{-1}w_1$,
$= 2v_1 = Rv_1$

10 Find $R, T \in \mathcal{L}(\mathbf{F}^4)$ such that R and T each have 2, 6, 7 as eigenvalues, R and T have no other eigenvalues, and there does not exist an invertible operator $S \in \mathcal{L}(\mathbf{F}^4)$ such that $R = S^{-1}TS$.

here, $\dim \mathbb{F}^4 = 4$, so some eigenspaces have dim 2
let $\dim E(2,R) = 2$ and $\dim E(6,T) = 2$
then, $E(2,T) = \text{null}(T-2I) \not\cong \text{null}(S^{-1}TS - 2I) = E(2, S^{-1}TS)$
since the eigenspace for the same eigenvalue in different operators may not be equal

14 (a) Give an example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.
(b) Suppose $\mathbf{F} = \mathbf{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

a) let $M(T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$T^2 = 0 \Rightarrow p(T) = 0$ with $p(z) = z^2$ with deg 2
$T \neq cI$, so deg 1 does not work.
Since $p(z) = z^2$ has repeated linear factors, T is not diagonalizable

21 The *Fibonacci sequence* $F_0, F_1, F_2, \dots$ is defined by
$F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 2$.
Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(x,y) = (y, x+y)$.
(a) Show that $T^n(0,1) = (F_n, F_{n+1})$ for each nonnegative integer n.
(b) Find the eigenvalues of T.
(c) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.
(d) Use the solution to (c) to compute $T^n(0,1)$. Conclude that
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$
for each nonnegative integer n.
(e) Use (d) to conclude that if n is a nonnegative integer, then the Fibonacci number F_n is the integer that is closest to
$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$
Each F_n is a nonnegative integer, even though the right side of the formula in (d) does not look like an integer. The number
$$\frac{1+\sqrt{5}}{2}$$
is called the ***golden ratio****.*

a) $T(x,y) = (y, x+y)$
$n=1$: $T(0,1) = (1,1) = (F_0, F_1)$
if $T^{n-1}(0,1) = (F_{n-1}, F_n)$, then
$T^n(0,1) = T(F_{n-1}, F_n) = (F_n, F_{n-1} + F_n)$
$= (F_n, F_{n+1})$ as required
b) $M(T) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $M(T^2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
Thus for $p(T) = 0$, $p(z) = T^2 + bT + cI$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
so, $0 = 1 + c \Rightarrow c = -1$
$0 = 1 + b \Rightarrow b = -1$
Thus $p(z) = z^2 - z - 1$
factors are $z = \frac{1 \pm \sqrt{5}}{2}$. Thus $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$

c) $\lambda_1 = \frac{1+\sqrt{5}}{2}$. find vectors in $\text{null}(T - \lambda I)$
$M(T - \lambda I) = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1+\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$
Find v s.t. $(T - \lambda I)v = 0$
$1x + \frac{1-\sqrt{5}}{2}y = 0$. $x = \frac{1-\sqrt{5}}{2}$, $y = -1$
$M(T - \lambda_2 I) = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}$
$1x + \frac{1+\sqrt{5}}{2}y = 0$, $x = \frac{1+\sqrt{5}}{2}$, $y = -1$
Basis $= \left\{\left(\frac{1-\sqrt{5}}{2}, -1\right), \left(\frac{1+\sqrt{5}}{2}, -1\right)\right\}$

d) $0 = a\left(\frac{1-\sqrt{5}}{2}\right) + b\left(\frac{1+\sqrt{5}}{2}\right) \Rightarrow a = \frac{1}{\sqrt{5}}$, $b = \frac{-1}{\sqrt{5}}$
$= \frac{1}{\sqrt{5}}\lambda_1 v_1 - \frac{1}{\sqrt{5}}\lambda_2 v_2$
now, $T^n(0,1) = \frac{1}{\sqrt{5}}T^n\lambda_1 v_1 - \frac{1}{\sqrt{5}}T^n\lambda_2 v_2 = (F_n, F_{n+1})$
$= \frac{1}{\sqrt{5}}\lambda_1\lambda_1^n v_1 - \frac{1}{\sqrt{5}}\lambda_2\lambda_2^n v_2$
$= \frac{1}{\sqrt{5}}\lambda_1^{n+1} v_1 - \frac{1}{\sqrt{5}}\lambda_2^{n+1} v_2$
$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\left(\frac{1-\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\left(\frac{1+\sqrt{5}}{2}\right)$
$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$
$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

e) skipped

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2 Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagonalizable. Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if every pair of elements of $\mathcal{E}$ commutes.
This exercise extends 5.76, which considers the case in which $\mathcal{E}$ contains only two elements. For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

$\Rightarrow$: Suppose $T_i, T_j \in \mathcal{E}$ s.t. with $B_V = \{v_1, \dots, v_n\}$, $M(T_i)$ and $M(T_j)$ are diagonal
Since all diagonal matrices commute, $M(T_i)M(T_j) = M(T_j)M(T_i)$, thus $T_iT_j = T_jT_i \ \forall i,j$
$\Leftarrow$: We know that $V = E(\lambda_1,T_1) \oplus \cdots \oplus E(\lambda_m,T_1)$
Since T_1 is diagonalizable, has eigenspace $E(\lambda_i,T_1)$, and commutes with T_2, then $E(\lambda_i,T_1)$ is invariant under T_2.
Since T_2 is diagonalizable, then $T_2|_{E(\lambda_i,T_1)}$ is diagonalizable. Consider μ be an eigenvalue in the restriction, then $E\left(\mu, T_2|_{E(\lambda_i,T_1)}\right) = E(\lambda_i,T_1) \cap E(\mu, T_2)$
Since T_3 is diagonalizable, then $T_3|_{E(\mu,T_2|_{E(\lambda_i,T_1)})}$ too. Thus, eigenvalue ν of $T_3|_{E(\mu,T_2|_{E(\lambda_i,T_1)})}$, then $E(\nu, T_3) = E(\lambda_i,T_1) \cap E(\mu,T_2) \cap E(\nu,T_3)$
Eventually, $\cup(\lambda_i, \mu_i, \dots, \omega) = E(\lambda_i,T_1) \cap E(\mu,T_2) \cap \cdots \cap E(\omega,T_m,..)$ is a basis for eigenvalues of $T_1, \dots, T_m$, so this basis diagonalizes $T_1, \dots, T_m$.

3 Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.
(a) Prove that $\text{null}\,p(S)$ is invariant under T.
(b) Prove that range $p(S)$ is invariant under T.
See 5.18 for the special case $S = T$.

a) let $v \in \text{null}\,p(S)$, i.e., $p(S)v = 0$. WTS that $pSTv = 0$
write $p(z) = a_kz^k + \dots + a_1z + a_0$
$p(S)T = (a_kS^k + \dots + a_1S + a_0I)T$
$= a_kS^kT + \dots + a_1ST + a_0T$
since S, T commute, $= a_kTS^kv + \dots + a_1TSv + a_0Tv$
$= T(a_kS^kv + \dots + a_1Sv + a_0Iv) = Tp(S)$
Since $p(S)$ and T commute, $p(S)(Tv) = (p(S)T)(v) = Tp(S)v = T \cdot 0 = 0$, thus $\text{null}\,p(S)$ is T-invariant
b) let $v \in \text{range}\,(p(S))$, WTS that $Tv \in \text{range}(p(S))$
$v = p(S)u$ for some $u \in V$.
$Tv = Tp(S)u = p(S)Tu$, so $Tv \in \text{range}\,p(S)$, thus range$(p(S))$ is T-invariant

5 Prove that a pair of operators on a finite-dimensional vector space commute if and only if their dual operators commute.
See 3.118 for the definition of the dual of an operator.

$\Rightarrow$ let $S, T \in \mathcal{L}(V)$ where $\dim V = n$
Suppose $ST = TS$. then $\forall \varphi \in V'$:
$(S'T')\varphi = S'(T'\varphi) = S'(\varphi \circ T) = \varphi \circ T \circ S = \varphi \circ (TS) = \varphi \circ (ST) = \varphi \circ S \circ T = T'(\varphi \circ S) = T'(S'\varphi)$, thus $S'T' = T'S'$
$\Leftarrow$ $S'T' = T'S'$, thus $S'T' - T'S' = 0 \Rightarrow (TS - ST)' = TS - ST \Rightarrow ST = TS$

9 Suppose V is a finite-dimensional nonzero complex vector space. Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.
(a) Prove that there is a vector in V that is an eigenvector for every element of $\mathcal{E}$.

Skipped

10 Give an example of two commuting operators S, T on a finite-dimensional real vector space such that $S + T$ has an eigenvalue that does not equal an eigenvalue of S plus an eigenvalue of T and ST has an eigenvalue that does not equal an eigenvalue of S times an eigenvalue of T.

Skipped

Inner Products And Norms

Examples:

- $\langle (w_1,\dots,w_n),(z_1,\dots,z_n)\rangle = w_1\overline{z}_1+\dots+w_n\overline{z}_n$
- $\langle f,g\rangle$ on $[-1,1] = \int_{-1}^{1} fg$
- $\langle p,q\rangle$ on $\mathcal{P}(\mathbb{R}) = p(0)q(0)+\int_{-1}^{1} p'q'$

Definitions:

- Dot product: $x\cdot y = x_1y_1+\dots+x_ny_n$
- $\|z\| = \sqrt{|z_1|^2+\dots+|z_n|^2}$. $\|z\|^2 = z_1\overline{z}_1+\dots+z_n\overline{z}_n = \langle z,z\rangle$. $\|\lambda v\| = |\lambda|\,\|v\|$
- Inner Product: on V is a function that takes each ordered pair (u,v) of elements of V to a number $\langle u,v\rangle \in \mathbb{F}$, and:
  - i) $\langle v,v\rangle \geq 0$
  - ii) $\langle v,v\rangle = 0$ iff $v=0$
  - iii) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$
  - iv) $\langle \lambda u,v\rangle = \lambda\langle u,v\rangle$
  - v) $\langle u,v\rangle = \overline{\langle v,u\rangle}$

  Also,
  - i) for fixed $v\in V$, $f: u\in V \to \langle u,v\rangle$ is a linear map $V\to\mathbb{F}$
  - ii) $\langle 0,v\rangle = 0$, $\langle v,0\rangle = 0$
  - iii) $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$
  - iv) $\langle u,\lambda v\rangle = \overline{\lambda}\langle u,v\rangle$
- Inner Product Space: is a V.S. V along with an inner product on V
- Orthogonal: u,v orthogonal if $\langle u,v\rangle = 0$

Propositions/Theorems:

- 6.11: 0 is orthogonal to all $v\in V$ and the only vector orthogonal to itself
- 6.12: $\|u+v\|^2 = \|u\|^2+\|v\|^2$
- 6.13 Orthogonal Decomposition: $u,v\in V$, $v\neq 0$, set $c = \frac{\langle u,v\rangle}{\|v\|^2}$ and $w = u-cv$, then $u = cv+w$ and $\langle w,v\rangle = 0$
- 6.14 Cauchy-Schwartz: $|\langle u,v\rangle| \leq \|u\|\,\|v\|$
- 6.17 Triangle Inequality: $\|u+v\| \leq \|u\|+\|v\|$
- 6.21 Parallelogram Equality: $\|u+v\|^2+\|u-v\|^2 = 2(\|u\|^2+\|v\|^2)$

Exercise 6.A.1. Prove or give a counterexample: If $v_1,\dots,v_m \in V$, then

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle v_j,v_k\rangle \geq 0.$$

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle v_j,v_k\rangle = \Big\langle \sum_{j=1}^{m} v_j, \sum_{k=1}^{m} v_k\Big\rangle = \Big\|\sum_{j=1}^{m} v_j\Big\|^2 \geq 0$$

Exercise 6.A.2. Suppose $S\in\mathcal{L}(V)$. Define $\langle\cdot,\cdot\rangle_S$ by

$$\langle u,v\rangle_S = \langle Su, Sv\rangle$$

for all $u,v\in V$. Show that $\langle\cdot,\cdot\rangle_S$ is an inner product on V if and only if S is injective.

Skipped

Exercise 6.A.4. Suppose $T\in\mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v\in V$. Prove that $T-\sqrt{2}I$ is injective.

Suppose $T-\sqrt{2}I$ is not injective, then $\exists v\neq 0$ s.t. $Tv = \sqrt{2}v$. WTS that $\|Tv\| > \|v\|$ $\forall v\in V$.

$\|Tv\| = \|\sqrt{2}v\| = \sqrt{2}\|v\| > \|v\|$ as required

Exercise 6.A.5. Suppose V is a real inner product space.

(a) Show that $\langle u+v, u-v\rangle = \|u\|^2-\|v\|^2$ for every $u,v\in V$.

(b) Show that if $u,v\in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.

(c) Use (b) to show that the diagonals of a rhombus are perpendicular to each other.

a) $\langle u+v,u-v\rangle = \langle u,u-v\rangle + \langle v,u-v\rangle$

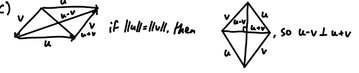
$= \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle$

$= \langle u,u\rangle - \langle v,v\rangle$

$= \|u\|^2 - \|v\|^2$ as required

b) if $\|u\| = \|v\|$, then $\langle u+v,u-v\rangle = \|u\|^2-\|v\|^2 = 0$

thus, $u+v$ is orthogonal to $u-v$ by definition

c) 

if $\|u\| = \|v\|$, then

, so $u-v \perp u+v$

Exercise 6.A.7. Suppose $u,v\in V$. Prove that $\|au+bv\| = \|bu+av\|$ for all $a,b\in\mathbf{R}$ if and only if $\|u\| = \|v\|$.

$\|au+bv\|^2 = a^2\|u\|^2+2ab\langle u,v\rangle + b^2\|v\|^2$ should equal $b^2\|u\|^2+2ba\langle u,v\rangle + a^2\|v\|^2$

$a^2\|u\|^2 - a^2\|v\|^2 + b^2\|v\|^2 - b^2\|u\|^2 = 0$

$(a^2-b^2)(\|u\|^2-\|v\|^2) = 0$, iff $\|u\|^2 = \|v\|^2$

Exercise 6.A.9. Suppose $u,v\in V$ and $\|u\| = \|v\| = 1$ and $\langle u,v\rangle = 1$. Prove that $u=v$.

WTS that $u=v$, or $u-v=0$

$\|u-v\|^2 = \langle u-v,u-v\rangle = \langle u,u\rangle - \langle v,u\rangle - \langle u,v\rangle + \langle v,v\rangle = \|u^2\| + \|v^2\| - 2\langle u,v\rangle = 1+1-2\cdot 1 = 0$

thus, $0 = \|u-v\|^2 = \|u-v\| = u-v \Rightarrow u=v$ as required

Exercise 6.A.11. Find vectors $u,v\in\mathbf{R}^2$ such that u is a scalar multiple of $(1,3)$, v is orthogonal to $(1,3)$, and $(1,2) = u+v$.

Decompose $x=(1,2)$, let $y=(1,3)$, find u,v s.t. $\langle x-cy,y\rangle = 0 \Rightarrow \langle x,y\rangle - c\langle y,y\rangle \Rightarrow c = \frac{\langle x,y\rangle}{\|y^2\|}$

$c = \frac{\langle (1,2),(1,3)\rangle}{\|(1,3)\|^2} = \frac{1\cdot 1+2\cdot 3}{1+9} = \frac{7}{10}$. so $u = cy = \frac{7}{10}(1,3) = \left(\frac{7}{10},\frac{21}{10}\right)$. $v = (1,2) - \left(\frac{7}{10},\frac{21}{10}\right) = \left(\frac{3}{10},\frac{-1}{10}\right)$

Check: $\langle v,y\rangle = \left\langle \left(\frac{3}{10},\frac{-1}{10}\right),(1,3)\right\rangle = \frac{3}{10}-\frac{3}{10} = 0$ and $u+v = \left(\frac{7}{10},\frac{21}{10}\right)+\left(\frac{3}{10},\frac{-1}{10}\right) = (1,2)$ as required

Exercise 6.A.12. Suppose a,b,c,d are positive numbers.

(a) Prove that $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16$.

(b) For which positive numbers a,b,c,d is the inequality above an equality?

Skipped

Exercise 6.A.21. Suppose $u,v\in V$ are such that

$$\|u\| = 3, \quad \|u+v\| = 4, \quad \|u-v\| = 6.$$

What number does $\|v\|$ equal?

$\|u+v\|^2+\|u-v\|^2 = 16+36 = 52 = 2(\|u\|^2+\|v\|^2) = 2(9+\|v\|^2)$

$52 = 2(9+\|v\|^2)$

$26 = 9+\|v\|^2$

$17 = \|v\|^2$

$\|v\| = \sqrt{17}$

Exercise 6.A.32. Suppose that E is a subset of V with the property that $u,v\in E$ implies $\frac{1}{2}(u+v)\in E$. Let $w\in V$. Show that there is at most one point in E that is closest to w. In other words, show that there is at most one $u\in E$ such that

$$\|w-u\| \leq \|w-x\|$$

for all $x\in E$.

Suppose u and v are closest to E, so $\|w-u\| = \|w-v\| = d$

from 6.A.31: $\|u-v\|^2/4 = \underbrace{\|w-u\|^2}_{d^2} - \underbrace{\|w-\frac{1}{2}(u+v)\|}_{\geq d^2} \leq 0$ since $\|w-u\| \leq \|w-\frac{1}{2}(u+v)^2\|$

so, $\|u-v\|^2 \leq 0$, but by def. $\|u-v\|^2 \geq 0$, $\Rightarrow \|u-v\| = 0$ so $u=v$

Orthonormal Bases:

Definitions:

- 6.22 Orthonormal: list of vectors are orthonormal if each vector has norm 1 and is orthogonal. i.e., $e_1,\dots,e_m$ in V are orthonormal if $\langle e_j, e_k\rangle = \begin{cases} 1, & \text{if } j=k \\ 0, & \text{if } j\neq k \end{cases}$
- 6.27 Orthonormal Basis: An orthonormal list in V also a basis of V

Propositions/Theorems:

- 6.24 If $e_1,\dots,e_m$ are orthonormal, then $\|a_1e_1+\dots+a_me_m\|^2 = |a_1|^2+\dots+|a_m|^2$ $\forall a_i\in\mathbb{F}$
- 6.25 Every orthonormal list of vectors is linearly independent
- 6.26 Bessel's Inequality: $e_1,\dots,e_m$ orthonormal. If $v\in V$, then $|\langle v,e_1\rangle|^2+\dots+|\langle v,e_m\rangle|^2 \le \|v\|^2$
- 6.28 Suppose $\dim V = n$. Every orthonormal list in V of length $\dim V$ is an orthonormal basis of V
- 6.30 Writ vector as l.c. of an orthonormal basis:
  - a) $v = \langle v,e_1\rangle e_1+\dots+\langle v,e_n\rangle e_n$
  - b) $\|v\|^2 = |\langle v,e_1\rangle|^2+\dots+|\langle v,e_n\rangle|^2$
  - c) $\langle u,v\rangle = \langle u,e_1\rangle\overline{\langle v,e_1\rangle}+\dots+\langle u,e_n\rangle\overline{\langle v,e_n\rangle}$
- 6.32 Gram-Schmidt Procedure: let $v_1,\dots,v_m$ be l.i. in V. Let $f_1=v_1$. For $k=2,\dots,m$, define f_k inductively by:
  $f_k = v_k - \frac{\langle v_k,f_1\rangle}{\|f_1\|^2}f_1 - \dots - \frac{\langle v_k,f_{k-1}\rangle}{\|f_{k-1}\|^2}f_{k-1}$. $\forall k$, let $e_k = \frac{f_k}{\|f_k\|}$. Then $e_1,\dots,e_m$ is orthonormal s.t. $\operatorname{span}(v_1,\dots,v_k)=\operatorname{span}(e_1,\dots,e_k)$
- 6.35 Every fin.dim inner prod. space has an orthonormal basis
- 6.36 Suppose V fin.dim. Every orthonormal list in V can be extended to orthonormal basis for V
- 6.37 $T\in\mathcal{L}(V)$ has up.-tri. matrix w.r.t some orthonormal basis of V iff the min. poly. of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1,\dots,\lambda_m\in\mathbb{F}$
- 6.38 Schur's theorem: Every operator on fin.dim complex inner product space has an up.-tri. matrix wrt some orthonormal basis
- 6.42 Riesz Representation Theorem: V fin.dim, φ lin.func. $\in V'$, then $\exists v\in V$ s.t. $\varphi(u)=\langle u,v\rangle$

Exercise 6.B.1. Suppose $e_1,\dots,e_m$ is a list of vectors in V such that
$$\|a_1e_1+\dots+a_me_m\|^2 = |a_1|^2+\dots+|a_m|^2$$
for all $a_1,\dots,a_m\in\mathbf{F}$. Show that $e_1,\dots,e_m$ is an orthonormal list.
This exercise provides a converse to 6.24.

take $a_k=1$ and $a_1,\dots,\hat{a_k},\dots,a_m=0$
then, $\|a_ke_k\|^2=|a_k|^2 \Rightarrow \|e_k\|=1$. Take $j\neq k$
$\|e_j\|^2=1\le 1+|a|^2=\|e_j+ae_k\|^2 \Rightarrow \langle e_j,e_k\rangle=0$
for arbitrary j,k. Thus, since e_j,e_k have norm 1
and are orthogonal, the list $e_1,\dots,e_m$ is orthonormal

Exercise 6.B.3. Suppose $e_1,\dots,e_m$ is an orthonormal list in V and $v\in V$. Prove that
$$\|v\|^2=|\langle v,e_1\rangle|^2+\dots+|\langle v,e_m\rangle|^2 \quad\Leftrightarrow\quad v\in\operatorname{span}(e_1,\dots,e_m).$$

$\Leftarrow$: $v\in\operatorname{span}(e_1,\dots,e_m)\Rightarrow v=a_1e_1+\dots+a_me_m$
$\forall k$, $\langle v,e_k\rangle=a_k$, thus $\|v\|^2=|a_1|^2+\dots+|a_m|^2=|\langle v,e_1\rangle|^2+\dots+|\langle v,e_k\rangle|^2$
$\Rightarrow$: $\|v\|^2=|\langle v,e_1\rangle|^2+\dots+|\langle v,e_m\rangle|^2$
let $u=\langle v,e_1\rangle e_1+\dots+\langle v,e_m\rangle e_m\in\operatorname{span}(e_1,\dots,e_m)$
$\|u\|^2=|\langle v,e_1\rangle|^2+\dots+|\langle v,e_m\rangle|^2=\|v\|^2$
$\langle u,v\rangle=\left\langle\sum_k^m\langle v,e_k\rangle,v\right\rangle=\sum_k^m\langle v,e_k\rangle\langle e_k,v\rangle=\sum|\langle v,e_k\rangle|^2=\|v\|^2$
now, $u=v\Rightarrow u-v=0\Rightarrow\|u-v\|^2=0$
$\|u-v\|^2=\|u\|^2-2\langle u,v\rangle+\|v\|^2=\|u\|^2-\|v\|^2=0\Rightarrow u=v\in\operatorname{span}(e_1,\dots,e_m)$

Exercise 6.B.4. Suppose n is a positive integer. Prove that
$$\frac{1}{\sqrt{2\pi}},\frac{\cos x}{\sqrt{\pi}},\frac{\cos 2x}{\sqrt{\pi}},\dots,\frac{\cos nx}{\sqrt{\pi}},\frac{\sin x}{\sqrt{\pi}},\frac{\sin 2x}{\sqrt{\pi}},\dots,\frac{\sin nx}{\sqrt{\pi}}$$
is an orthonormal list of vectors in $C[-\pi,\pi]$, the vector space of continuous real-valued functions on $[-\pi,\pi]$ with inner product
$$\langle f,g\rangle=\int_{-\pi}^{\pi}fg.$$
Hint: The following identities will be helpful.
skipped
$$(\sin x)(\cos y)=\frac{\sin(x-y)+\sin(x+y)}{2}$$
$$(\sin x)(\sin y)=\frac{\cos(x-y)-\cos(x+y)}{2}$$
$$(\cos x)(\cos y)=\frac{\cos(x-y)+\cos(x+y)}{2}$$

Exercise 6.B.6. Suppose $e_1,\dots,e_n$ is an orthonormal basis of V.
(a) Prove that if $v_1,\dots,v_n$ are vectors in V such that
$$\|e_k-v_k\|<\frac{1}{\sqrt{n}}$$
for each k, then $v_1,\dots,v_n$ is a basis of V.
(b) Show that there exist $v_1,\dots,v_n\in V$ such that
$$\|e_k-v_k\|\le\frac{1}{\sqrt{n}}$$
for each k, but $v_1,\dots,v_n$ is not linearly independent.
This exercise states in (a) that an appropriately small perturbation of an orthonormal basis is a basis. Then (b) shows that the number $1/\sqrt{n}$ on the right side of the inequality in (a) cannot be improved upon.

Exercise 6.B.7. Suppose $T\in\mathcal{L}(\mathbf{R}^2)$ has an upper-triangular matrix with respect to the basis $(1,0),(1,1)$, $(1,1),(1,2)$. Find an orthonormal basis of $\mathbf{R}^2$ with respect to which T has an upper-triangular matrix.

Exercise 6.B.9. Suppose $e_1,\dots,e_m$ is the result of applying the Gram-Schmidt procedure to a linearly independent list $v_1,\dots,v_m\in V$. Prove that $\langle v_k,e_k\rangle>0$ for each $k=1,\dots,m$.

Exercise 6.B.16. Suppose V is finite-dimensional. Suppose $\langle\cdot,\cdot\rangle_1,\langle\cdot,\cdot\rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists a positive number c such that $\|v\|_1\le c\|v\|_2$ for every $v\in V$.

Exercise 6.B.19. Suppose $v_1,\dots,v_n$ is a basis of V. Prove that there exists a basis $u_1,\dots,u_n$ of V such that
$$\langle v_j,u_k\rangle=\begin{cases}0 & \text{if } j\neq k,\\ 1 & \text{if } j=k.\end{cases}$$

Orthogonal Complements and Minimization Problems

Definitions:

- **6.46:** If $U \subseteq V$, the orthogonal complement of U is $U^\perp = \{v \in V : \langle u, v\rangle = 0 \text{ for every } u \in U\}$
- **6.48:** Orthogonal complement properties:
  - i) $U \subseteq V$, then $U^\perp \subseteq V$
  - ii) $\{0\}^\perp = V$ and $V^\perp = \{0\}$
  - iii) $U \subseteq V$, then $U \cap U^\perp \subseteq \{0\}$
  - iv) if $G \subseteq V$, $H \subseteq V$, then if $G \subseteq V$, then $H^\perp \subseteq G^\perp$
- **6.55:** $U \subseteq V$, then $P_U \in \mathcal{L}(V)$ is orthogonal projection of V onto U.
  $\forall v \in V$, $v = u + w$, with $u \in U$ and $w \in U^\perp$. Then let $P_U v = u$
- **6.57:** Orthogonal projection properties:
  - i) $P_U \in \mathcal{L}(V)$
  - ii) $P_U u = u \ \forall u \in U$. $P_U w = 0 \ \forall w \in U^\perp$
  - iii) range $P_U = U$. null $P_U = U^\perp$
  - iv) $v - P_U v \in U^\perp \ \forall v \in V$
  - v) $P_U^2 = P_U$
  - vi) $\|P_U v\| \leq \|v\| \ \forall v \in V$
  - vii) if $e_1, \ldots, e_m$ is an orthonormal basis of U and $v \in V$, then $P_U v = \langle v, e_1\rangle e_1 + \ldots + \langle v, e_m\rangle e_m$
- **6.68:** $T \in \mathcal{L}(V, W)$. Pseudoinverse $T^\dagger \in \mathcal{L}(V, W)$ is $T^\dagger w = (T|_{(\text{null } T)^\perp})^{-1} P_{\text{range } T}\, w \ \forall w \in W$

Propositions/Theorems:

- **6.49:** $V = U \oplus U^\perp$
- **6.51:** $\dim U^\perp = \dim V - \dim U$
- **6.52:** $U = (U^\perp)^\perp$
- **6.54:** $U^\perp = \{0\} \iff U = V$
- **6.61:** $U \subseteq V$, $v \in V$, $u \in U$, then $\|v - P_U v\| \leq \|v - u\|$.
  Also, $\|v - P_U v\| = \|v - u\|$ iff $u = P_U v$
- **6.67:** $T \in \mathcal{L}(V, W)$, then $T|_{(\text{null } T)^\perp}$ is an injective map $(\text{null } T)^\perp \to \text{range } T$
- **6.69:** Pseudoinverse algebraic properties:
  - i) if T^{-1} exists, then $T^\dagger = T^{-1}$
  - ii) $TT^\dagger = P_{\text{range } T}$ = the orthogonal projection of W onto range T
  - iii) $T^\dagger T = P_{(\text{null } T)^\perp}$ = the orthogonal projection of V onto $(\text{null } T)^\perp$
- **6.70:** Pseudoinverse provides best approximation solution or best solution
  Suppose V finidm, $T \in \mathcal{L}(V, W)$, $w \in W$:
  - a) If $v \in V$, then $\|T(T^\dagger w) - w\| \leq \|Tv - w\|$ with equality iff $v \in T^\dagger w + \text{null } T$
  - b) If $v \in T^\dagger w + \text{null } T$, then $\|T^\dagger w\| \leq \|v\|$ with equality iff $v = T^\dagger w$

HW

Self-Adjoint and Normal Operators

Definitions:

7.1 Adjoint: $T \in \mathcal{L}(V,W)$. The adjoint of T is $T^*: W \to V$ s.t. $\langle Tv, w\rangle = \langle v, T^*w\rangle$ $\forall v \in V, \forall w \in W$

7.5: Properties of the adjoint: Suppose $T \in \mathcal{L}(V,W)$. Then:

i) $(S+T)^* = S^* + T^*$

ii) $(\lambda T)^* = \bar{\lambda} T^*$ $\forall \lambda \in \mathbb{F}$

iii) $(T^*)^* = T$

iv) $(ST)^* = T^* S^*$ $\forall S \in \mathcal{L}(W,U)$, with U finite IPS

v) $I^* = I$

vi) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$

7.6: Suppose $T \in \mathcal{L}(V,W)$. Then:

i) $\text{null } T^* = (\text{range } T)^\perp$

ii) $\text{range } T^* = (\text{null } T)^\perp$

iii) $\text{null } T = (\text{range } T^*)^\perp$

iv) $\text{range } T = (\text{null } T^*)^\perp$

7.7 Conjugate transpose: $(A^*)_{jk} = \overline{A_{kj}}$ where $A \in \mathbb{F}^{m,n}$

7.10 Self-adjoint: An operator $T \in \mathcal{L}(V)$ is called self-adjoint if $T = T^*$

7.18 Normal: $T \in \mathcal{L}(V)$ is normal if $TT^* = T^*T$, and every self-adjoint operator is normal.

Propositions/Theorems:

7.4: If $T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$

7.9: Let $T \in \mathcal{L}(V,W)$. $B_V = \{e_1, \dots, e_n\}$ orthonormal and $B_W = \{f_1, \dots, f_m\}$ orthonormal.
$M(T^*, B_W, B_V) = M(T, B_V, B_W)$. In other words, $M(T^*) = (M(T))^*$

7.12: Every eigenvalue of a self-adjoint operator is real
→ Let $Tv = \lambda v$
$\langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\|v\|^2$
$\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\|v\|^2$
Thus $\lambda = \bar{\lambda}$, so λ is real

7.13: Assuming $\mathbb{F} = \mathbb{C}$, $\langle Tv, v\rangle = 0 \Leftrightarrow T = 0$

7.14: Assuming $\mathbb{F} = \mathbb{C}$, $\langle Tv, v\rangle$ is real $\forall v \in V \Leftrightarrow T$ is self-adjoint

7.16: T self-adjoint and $\langle Tv, v\rangle = 0$ $\forall v \Leftrightarrow T = 0$

7.20: T is normal $\Leftrightarrow \|Tv\| = \|T^*v\|$ $\forall v \in V$
→ $\|Tv\|^2 = \langle Tv, Tv\rangle = \langle T^*Tv, v\rangle$
$\|T^*v\|^2 = \langle T^*v, T^*v\rangle = \langle TT^*v, v\rangle$
By normal, $T^*T - TT^* = 0 \Leftrightarrow \langle (TT^* - T^*T)v, v\rangle = 0$ by props
$\Leftrightarrow \|T^*v\|^2 - \|Tv\|^2 = 0 \Rightarrow \|T^*v\| = \|Tv\|$

7.21: Suppose $T \in \mathcal{L}(V)$ is normal, then:

i) $\text{null } T = \text{null } T^*$, $\text{range } T = \text{range } T^*$

ii) $V = \text{null } T \oplus \text{range } T$

iii) $T - \lambda I$ is normal $\forall \lambda \in \mathbb{F}$

iv) if $v \in V$, $\lambda \in \mathbb{F}$, then $Tv = \lambda v \Leftrightarrow T^*v = \bar{\lambda} v$

7.22: Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal

7.23: Suppose $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. T is normal $\Leftrightarrow$ there exist commuting self-adjoint operators A, B s.t. $T = A + Bi$

HW

Week 12

Definitions:

- $S \in \mathcal{L}(V,W)$ is called an isometry if $\|Sv\| = \|v\|$ $\forall v \in V$
  Sps $v \in \text{null}(S)$. $\|v\| = \|Sv\| = \|0\| = 0 \Rightarrow v = 0 \Rightarrow S$ injective
- $S \in \mathcal{L}(V)$ is unitary if it is an invertible isometry
- SVD: Let $T \in \mathcal{L}(V,W)$. The singular values of T are the nonnegative square roots of the eigenvalues of T^*T, listed in decreasing order with repetition = $\dim E(\lambda, T^*T)$
  Equivalently, diagonalize $T^*T \in \mathcal{L}(V)$. The singular values of T are the square roots of diagonal entries of $M(T^*T)$

Propositions/Theorems:

- $T \in \mathcal{L}(V)$ is self-adjoint. Min poly of T splits into linear factors z^2+bz+c with $b^2 < 4c$ and $T^2+bT+cI$ is invertible
- Real Spectral Theorem: Let V be $\mathbb{R}$ v.s. and $T \in \mathcal{L}(V)$. TFAE:
  i) T is self adjoint
  ii) T is diagonalized by an orthonormal basis of V
  iii) V has an orthonormal basis of eigenvectors of T
- Complex Spectral Theorem: Let V be $\mathbb{C}$ v.s., $T \in \mathcal{L}(V)$. TFAE:
  i) T is normal
  ii) T is diagonalized by an orthonormal basis of V
  iii) V has an orthonormal basis of eigenvectors of T
- Isometry Properties: Let $S \in \mathcal{L}(V,W)$. TFAE:
  i) S is an isometry
  ii) $S^*S = I$
  iii) $\langle Su, Sv \rangle = \langle u, v \rangle$ $\forall u,v$
  iv) (V, W findim). Let $M(S)$ wrt orthonormal basis of V and W. Then $M(S)$ has as its cols orthonormal vectors wrt the dot product
- Unitary Properties: Let $T \in \mathcal{L}(V)$. TFAE:
  i) T is unitary
  ii) T is unitary iff T^* is unitary
  iii) $TT^* = T^*T = I$ $(T^{-1} = T^*)$
- Let $S \in \mathcal{L}(V)$ unitary. λ eigenvalue of S. Then $|\lambda| = 1$
  $\|Sv\| = \|v\| = \|\lambda v\| = |\lambda| \cdot \|v\| \Rightarrow |\lambda| = 1$
- V f.d. $\mathbb{C}$ v.s., $S \in \mathcal{L}(V)$. S is unitary iff there is an orthonormal basis of V w/ e-vecs of S that have e-vals w/ abs 1.
- Prop: $T \in \mathcal{L}(V,W)$. Then:
  a) T^*T is a "positive operator": self-adjoint w/ nonneg values
  b) $\text{Null}(T^*T) = \text{Null}(T)$
  c) $\text{Range}(T^*T) = \text{Range}(T^*)$
  d) $\dim \text{range } T = \dim \text{range } T^* = \dim \text{range } T^*T$
- Prop: Let $T \in \mathcal{L}(V,W)$. Then:
  a) T is injective iff 0 is not a SV of T
  b) $\dim \text{range } T$ = # of positive SVs
  c) T is surjective iff it has $\dim W$ positive SVs
- Prop: Let $S \in \mathcal{L}(V,W)$. S is an isometry ($\langle Sv, Su \rangle = \langle v, u \rangle$) iff SVs = 1
- SVD: Let $T \in \mathcal{L}(V,W)$. Let $(s_1, \dots, s_m)$ be positive SVs of T.
  Then there are orthonormal vectors $(e_1, \dots, e_m) \in V$ and $(f_1, \dots, f_m) \in W$ s.t. for any $v \in V$,
  $Tv = s_1\langle v, e_1 \rangle f_1 + \dots + s_m \langle v, e_m \rangle f_m$
- Corollary: Let T, s_k as above. Take $e_1, \dots, e_m$ to be full ON basis of V.
  Also extend $f_1, \dots, f_m$ to ON basis $f_1, \dots, f_p$ of W. Then $Te_k = s_k f_k$. So wrt these bases,
  $M(T)_{jk} = s_k$ if $1 \le j = k \le m$, or 0 otherwise.
- Corollary: Suppose $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. $A = M(T)$ wrt standard basis.
  B: $m \times m$: $B = (f_1, \dots, f_m)$
  C: $n \times m$: $C = (e_1, \dots, e_m)$
  D: $m \times n$: $D = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_m \end{pmatrix}$
  Then, $A = BDC^*$

Jordan Normal Form:

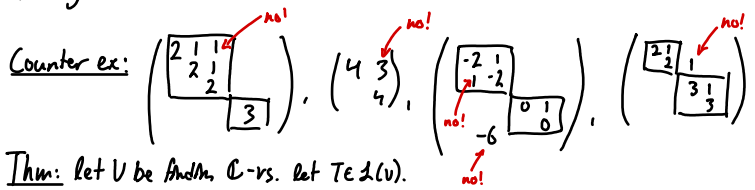
Def: Let $T \in \mathcal{L}(V)$. $v_1, \dots, v_m$ a basis of V. Basis is a jordan basis of V for T and $M(T)$ is in Jordan normal form: $\begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ & & & A_n \end{pmatrix}$ where $A_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}$ (matrices $\to A_n$)

Ex:



All diagonal matrices are in Jordan normal form

Counter ex:



Thm: Let V be findim, $\mathbb{C}$-vs. Let $T \in \mathcal{L}(V)$.
Then there is a Jordan basis for T.